Modelling Stock Prices

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CHAPTER 5: Modelling stock prices

Overview

Keywords - Economy:
- Stock market models
- Arbitrage
- Trading strategy
- Binomial model
- Black-Scholes-model

Keywords – Elementary mathematics:
- Binomial distribution
- Normal distribution
- Bar plot (histogram)
- Exponential function
- Natural logarithm
- Continuous functions

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Chapter 5 guidelines

In this chapter explicit modelling of the development of stock prices in a time lapse will be carried out. In doing so we will, by using the binomial and the Black-Scholes model, develop stock market models which can be used in practice and which feature the irregular shape typical of stock prices. This, as far as the form is concerned, almost in no way resembles the functions that appear as part of conventional class instruction.

In order to be able to comprehend this chapter, basic knowledge of calculus of probabilities (as conveyed in e.g. Section 5.6) will be required. In particular the binomial and normal distribution will play a crucial part. It is indeed possible to do without sections which are based on normal distribution, yet this is not recommendable due to the meaning that these models have in practice.
In Section 5.1 we will firstly identify certain aspects and the necessity for explicit stock price modelling. In discussion sections 5.2/4/6 both the binomial and the Black-Scholes model will be introduced. Furthermore the economic concept of the no-arbitrage condition, fundamental to stock price modelling (and especially to option pricing, see Chapter 7) will be clearly transmitted. Character and shape of binomial and normal distribution represent the content of Section 5.3, which can be omitted, depending on the previous knowledge of the students. The problem of probability space, belonging to normal distribution, cannot be dealt with within the scope of this book. In this section there is also an opportunity to make the already experimental connection between normal and binomial distribution (see Discussion 1).

The term no-arbitrage condition, meaning that the possibility of gaining profit with no risk and equity investment is not available, will be precisely formulated in Section 5.5 by analyzing the case of a one-period model. Because this term has a central meaning – among other for the next chapter – it should be discussed in detail (e.g. as presented in Discussion 5.4). The formal introduction to the binomial model and the Black-Scholes model will occur in Section 5.7, by employing the de Moivre-Laplace theorem to show that the Black-Scholes model may be obtained as a continuous-time limit of the binomial model.

In Section 5.8 basic principles for simulating accidental events, and particularly stock prices, will be introduced. Here we are offered an opportunity to implement individual models by means of a computer. Generally in financial mathematics there is great importance attached to simulation, which will be expanded by the Monte Carlo method in the following chapter.

5.1 Observing the development of assets

A foresight

It would be wonderful to be able to perfectly foretell what the stock prices will be like, because in that case one could really strike optimal investment decisions. Unfortunately this is not possible due to manifold influences which determine the layout of a stock price (such as e.g. value of the future prospects of the company, general economic situation, political decisions, consumer behaviour, etc.). The first indications of the future development of a stock price can provide us with estimations for the expected value and the variance of the rate of return of a security. This rough model for the development of a stock price (employed in Chapter 4), which is always only oriented towards one single point in time in the future, is however not particularly helpful if we are dealing with a complicated problem. In such a case one needs a model which takes into consideration many points in time in the future or even a continuous development of security prices. In practice we often use a model called the geometrical Brownian movement for modelling stock prices, which we will approach more closely in this chapter. This model takes into consideration continuous stock price development. At the same time the continuity refers to the time modelling (the development of the price will be observed in all future points in time) as well as to the value development of the stock (i.e. it is assumed that the stock price is a continuous function of time). This does not completely correspond to reality because the prices change in leaps (however often in very small ones), yet this model has in the meantime proved itself in practice and is presently becoming more and more sophisticated concerning its application in real life. It is a very good aid when calculating exercise prices and detecting risk. By using this model one can also simulate capital development and in this way dare to venture a look into the possible outcome in the future.

Objective assessment of capital and risk

Nowadays banks play a very important part in the economy, among other as credit grantors and financial intermediaries, contractors of financial investment opportunities, as well as providing
other financial services. By fulfilling such tasks the banks are exposed to various types of risk. For example it can happen that a debtor suddenly cannot pay back his loan. However, financial losses arise in such a way that the international assets of a bank become almost valueless through strong devaluation of the appropriate currency. Additionally it can happen that the computer network collapses and the losses arise through erroneous entries and lost transactions. A bank can end up in trouble if due to a sudden incident (e.g. loss of a great debtor and therefore loss of trust in the bank) many customers clear their accounts at the same time (liquidity risk). However, all these risks should not lead to instability of the bank system. In accord with the fact that the savings of the customers should be protected, the reliable banks are equated with a good economic system. Accordingly they should give the people a feeling of certainty and stability. This is why banks in many countries all over the world are obligated to secure their loans and market price risk through equity capital (see the 1988 Basel Capital Accord and the 1996 Basel Market Risk Paper), which is monitored by the supervision of banking. New developments in the mathematical research have lead to more improved financial instruments and new methods of risk control. This is the reason why the directives from 1988 and 1996 meanwhile seem out-dated and the Basel Committee on Banking Supervision is currently working on new proposals for the international central banks, all of which can be found on the Internet under the password „Basel II“. Accordingly it will also be considered how to secure operational risks (e.g. computer crashes) in the future.

These decrees and the ones still to come mean that the financial institutes will continually have to have the exact knowledge on their capital values and the possible future development. This can motivate and force the banks in particular to spend a lot of time developing new mathematical security models and to configure them by adding further features even more realistically. In this way major banks hold for different markets (stock market, currency markets, option market, etc.) also often different models which are adjusted to the characteristics of the respective markets and by means of which they conduct simulations (see Section 5.8) or price calculations (see Section 6.4).

5.2 Discussion: New models in Düsseldorf

At the moment the conductor is bringing a tray with four cups of coffee and three croissants to the conference compartment of the train whizzing through the landscape at 250 km/h. Inside Selina, Oliver, Nadine and Sebastian from the Clever Consulting Team are sitting and preparing for a new assignment in Düsseldorf. The „Deutsche Kunst- und Kulturbank Inc.“, whose headquarters are situated in Düsseldorf, invited the Team to for once carefully test the mathematical market models which were developed for them by other management consultants.

Nadine: New market models in the Deutsche Kunst- und Kulturbank Inc., well, well! I've never heard a thing about this bank in my entire life.

Selina: It won't be around that much longer anyway. The gap in the market used by this bank was discovered only recently. It allocates loans to museums and interpretive centers and finances big rock concerts, such as for example the last big concert by the Green Mild Peppers in Köln. Furthermore it gives loans to perspective fashion designers and finances fashion shows.

Sebastian: Which is why the bank has its headquarters in Düsseldorf, the fashion capital.

Oliver: I think I now know the real reason why you, Selina, were just dying to go with us to see our new client. As far as I can remember you have very little idea about mathematical stock models!

Selina: Firstly you also need a neutral person with basic knowledge on business economy who can challenge your grey matter critically. Secondly you could now explain a thing or two to me
and in this way really gather momentum. Thirdly I'll be your best shopping consultant in the world if you want to refresh your business outfit after closing time.

*Oliver:* I could definitely use a fashion consultant. In exchange for that I'll gladly explain everything about stock price models. Besides I think that you'll listen attentively when it comes to predicting stock prices.

*Selina:* Predicting stock prices, even I don't believe in that. The prices are so strongly influenced by chance that you can at most indicate a hoped for long term *trend*.

*Sebastian:* But with an appropriate model you can at least make some important decisions. I will exemplify this now with a *one-period binomial model*.

*Selina:* This is an ancient financial maths theory. We need a continuous model for development of stock prices within the whole period of observation from today to the planning interval of an investment problem. Which one of us just passed on his croissant? Oliver, are you on a diet or something?

*Oliver:* For God's sake, no. It must be Selina who's trying to starve herself into her new model outfit. Aside from that, let's start with the simpler stock price model.

*Selina:* No, this *model simplifies reality* too strongly. In principle it only focuses on two points in time, namely today and one point in the future. And while we're doing that, the time runs continuously.

*Sebastian:* We can approximate that by creating a model and then observing many small periods...

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While the management consulting team is devouring their croissants, with one exception, and still quarrelling a bit about realistic and unrealistic models, as well as the appropriate beginning of the crash course, let us catch up on a few basic principles of mathematics.

**Discussion 1:**

At this point we can discuss modelling in general. One can further contemplate which simplifications one would suggest for a market model (e.g. only one stock, loan interest corresponds to fixed deposit interest, etc.). Which problems can arise if one does not simplify enough?

**5.3 Basic principles of mathematics: Binomial and normal distribution**

**A special case of binomial distribution: the Bernoulli distribution**

The simplest of all random distributions is arguably the *Bernoulli distribution* (named after the mathematician, Jakob Bernoulli, 1654-1705), a special case of binomial distribution. It describes random experiments during which something particular happens or does not happen. The random experiment therefore has two possible outcomes, which can be designated with either 1 ("it is happening" or also "success") or 0 ("it is not happening" or also "no success").

As an example we will observe a fair toss of a coin. A "1" can be assigned to the event "heads" and a "0" to the event "tails". Both outcomes have the same probability, i.e.

\[
P(\{1\}) = \frac{1}{2} = P(\{0\}).
\]

However, not in case of all random experiments with only two possible events do both outcomes have the same probability. One can e.g. consider the sex of a new-born child a random experi-
ment. Meanwhile many empirical studies have revealed that more boys than girls are brought to the world. This makes the probability of getting a boy a bit bigger than 1/2, assuming that \( P(\{\text{boy}\}) = 0.51 \).

According to the calculation rules for probability, it is true that \( P(\{\text{girl}\}) = 1 - 0.51 = 0.49 \).

One can write the Bernoulli experiment by indicating the probability of "success", thus the outcome "1":

\[
P(\{1\}) = p, \quad 0 \leq p \leq 1.
\]

The probability of the event "0" results then in

\[
P(\{0\}) = 1 - p.
\]

The expected value of the random variable \( X \) distributed according to Bernoulli, which only takes on the values 0 or 1, can be very easily calculated:

\[
E(X) = p \cdot 1 + (1 - p) \cdot 0 = p.
\]

We have the following result for variance:

\[
\text{Var}(X) = E(X^2) - [E(X)]^2 = 1^2 \cdot p + 0^2 \cdot (1 - p) - p^2 = p \cdot (1 - p)
\]

and in a summarized form we get:

\[
\text{Bernoulli distribution:} \quad \text{A random experiment is called the Bernoulli experiment if the experiment has only two possible outcomes which are designated with 1 or 0. It suffices to indicate the probability of "success" – the event "1":}
\]

\[
P(\{1\}) = p, \quad 0 \leq p \leq 1 \quad \Rightarrow P(\{0\}) = 1 - p.
\]

A random variable \( X \) distributed according to Bernoulli takes on only the values 0 or 1 and for such a variable it is true that:

\[
\text{Expected value: } E(X) = p, \quad \text{Variance: } \text{Var}(X) = p \cdot (1 - p).
\]

(\(\rightarrow\) Ex.5.1, Ex.5.2)

The binomial distribution

If the same Bernoulli experiment is conducted more than once independently in a sequence and we count the number of experiments in which our special case, which we assigned as "1", is existent, the random variable "number" exhibits the binomial distribution.

A simple example of this is to try and toss the same coin three times one after the other and count how many times we tossed heads. Or we can observe a family with five children of different age (meaning no twins because in case of enzygotic twins the sex of one depends on the sex of the other), and count the number of girls.

\[
\text{Binomial distribution:} \quad \text{If the Bernoulli experiment is conducted with success probability } p \text{ \( n \)-times consecutively, it is true for the random variable } X \text{, which counts the number of successes:}
\]

\[
P(\{X = k\}) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{(n-k)}, \quad 0 \leq k \leq n.
\]
We then also state that the random variable $X$ is distributed binomially with parameters $0 < p < 1$ and $n \in \mathbb{N}$ and write this as

$$X \sim B(n, p).$$

Such a binomially distributed random variable can thus only take on the values $0, 1, \ldots, n$ and it is true that:

\[
\text{Expected value: } E(X) = n \cdot p, \quad \text{Variance: } Var(X) = n \cdot p \cdot (1 - p).
\]

The probability $P(X=k)$ for $k$ successes can be best noted by realizing that $p^k \cdot (1-p)^{n-k}$ is the probability for a firm sequence of $n$ independent Bernoulli experiments with $k$ successes and $n-k$ failures. The coefficient $n!/(k!(n-k)!)$ (i.e. the binomial coefficient) indicates in how many ways one can arrange $k$ successes and $n-k$ failures in $n$ experiments.

The probability of tossing heads only once when tossing a coin fairly three times then amounts to

$$P(X=1) = \frac{3!}{1! \cdot (3-1)!} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^2 = 0.375,$$

therefore a bit more than a third.

For a family with five children and no twins one can calculate the probability of having girls as

$$P(X=5) = \frac{5!}{5! \cdot (5-5)!} \cdot 0.49^5 \cdot 0.51^0 = 0.028,$$

meaning that this occurs only at the probability of less than $3\%$.

Due to the fact that the random variable $X$ belonging to the binomial experiment in which we count the number of successes is created by adding independent random variables $X_1, \ldots, X_n$ of the individual Bernoulli experiments, the expected value and variance of $X = X_1 + \ldots + X_n$

can be very easily calculated:

\[
E(X) = E(X_1) + \ldots + E(X_n) = n \cdot p,
\]
\[
Var(X) = Var(X_1) + \ldots + Var(X_n) = n \cdot p \cdot (1 - p).
\]

One has to pay attention that the covariance between $X_i, i=1,\ldots,n$ is zero because we assume the independent random variables.

($\rightarrow\text{Ex.5.3, Ex.5.4}$)

**The normal distribution**

Without exaggeration one can arguably assume that the normal distribution (also designated as the Gauß distribution) is the most important of all random distributions. One can notice it everywhere in everyday life. However it exhibits (considered within the scope of this book) one peculiarity: The set $\Omega$ of all possible outcomes of the belonging random experiment is not finite, in fact $\Omega$ encompasses even all real numbers. The consequence of this is that not all possible values with positive probability can be adopted. Moreover, as a matter of fact not a single of these values with positive probability is adopted! This means that we need the term probability density, on which we will elaborate shortly.

It is best to observe the normal distribution on the basis of examples:
- The body height of all 20-year-old women living in one city is approximately distributed normally.
- If a class should guess the unknown width of a desk, the estimated values of the students are roughly normally distributed.
- The weight of 6-weeks-old male white mice is approximately normally distributed.
- A small workpiece is measured by different people with a precision measuring device. Despite the accuracy there are measuring errors. These are then approximately normally distributed.
- On a New Year’s Eve a party crowd tries to foretell the price of a specific type of champagne for the next end of the year. On New Year’s Eve next year they realize that the estimate error (correct price – estimated price) is roughly normally distributed.

All the examples have something in common: There is an average value named the “accurate value”, or a type of normal value around which all the other values are distributed. In fact the other values distribute themselves symmetrically all around this central value. There are approximately equally many values that are positioned beneath it as there are those that are above it. Most of the values are situated in the proximity of this central value, and the values which are situated very far away appear very rarely.

If one draws up a bar plot (histogram), as a result one gets a picture which could look similar to the following one (based on 200 random data):

![Chart 5.1](image)

*Chart 5.1 Histogram of a random sample of normally distributed random variables*

If one now evaluated more and more random data and arranged the categories of the bar plot always more precisely, one would get a bell-shaped form which bears similarity to the adjacent drawing. This drawing displays the density of the standard normal distribution $\phi(x)$:
In the probability theory **density** represents a non-negative real function which in principle models an "ideal bar plot". By means of density we can calculate probability. This is due to the fact that the surface area beneath the curve of the density from \( y \) to \( z \) indicates directly the probability that in the associated random experiment a value from the interval \([y, z]\) is adopted. Consequently the entire area between the \( x \)-axis and density must have the content of one.

**Definition:**
A random variable \( X \) has **density** \( f : \mathbb{R} \rightarrow [0, \infty) \) if it is true for all values \( y \leq z \) with \( y, z \in \mathbb{R} \) that

\[
P\left(\{y \leq X \leq z\}\right) = \int_{y}^{z} f(x) \, dx.
\]

For the expected value of random variables \( X \) it is then true that

\[
E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx,
\]

if this value is finite.

This results in

\[
P\left(\{X = y\}\right) = \int_{y}^{y} f(x) \, dx = 0.
\]

In this case one has to pay attention that a normally distributed random variable can adopt all possible real values. It is so improbable that this particular value \( y \) is exactly specified, that the
probability of zero is always assigned to one single value. Despite of that, the value of density tells us something about probability with which the \( y \) is (almost) specified. If the density \( f(.) \) is namely continuously in \( y \), for small values \( \varepsilon > 0 \) it is thus true that

\[
P\left( \{ y - \varepsilon \leq X \leq y + \varepsilon \} \right) = \int_{y-\varepsilon}^{y+\varepsilon} f(x) \, dx = 2 \cdot \varepsilon \cdot f(y).
\]

Therefore, the bigger \( f(y) \) is, the more probable it is that \( X \) will take on the values in the proximity of \( y \). Based on the above-depicted examples one expects that in case of a normally distributed random variable the interval which contains the "normal value" has a bigger probability than the interval of the same length which is positioned far away from the "normal value". This is exactly what one observes in the density of the normal distribution, which adopts the highest value in the "normal value". Based on examples one can equally very well imagine that the interval ("normal value" - \( y \), "normal value" ] has the same probability as ["normal value", "normal value" + \( y \)]. This is also in accord with the density of the normal distribution, which is symmetrical.

**Normal distribution:**

a) The **density of the normal distribution** is expressed in the following way:

\[
\phi(x) = \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma^2}} e^{-\frac{(x-\mu)^2}{2 \sigma^2}}, \quad \sigma > 0, \quad \mu \in \mathbb{IR}.
\]

If \( \mu = 0 \) and \( \sigma = 1 \), one calls this the **density of the standard normal distribution**.

b) The **distribution function of the standard normal distribution** is expressed in the following manner:

\[
\Phi(z) = P\left( \{ X \leq z \} \right) = \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx.
\]

c) If the random variable \( X \) is normally distributed with parameters \( \mu \in \mathbb{IR} \) and \( \sigma > 0 \), one writes also:

\( X \sim N\left( \mu, \sigma^2 \right) \).

A normally distributed random variable can adopt the values from the entire \( \mathbb{IR} \) and it is true that:

Expected values: \( E(X) = \mu \), Variance: \( Var(X) = \sigma^2 \).

Unfortunately the integral cannot be explicitly calculated in the distribution function of the standard normal distribution. Due to the importance of this distribution, one can calculate and tabulate the integral and therefore also the \( \Phi(z) \) by using numerical methods for many values of \( z \). Such tables can be found in statistical books (e.g. Henze(1997)). Because of

\[
\Phi(-z) = 1 - \Phi(z),
\]

as a rule only the values of \( \Phi(z) \) are tabulated for positive \( z \). The function \( \Phi \) can be used to calculate the probability of intervals \((y, z]\) or \((z, \infty)\) of random variables distributed according to the standard normal distribution

\[
P\left( \{ y < X \leq z \} \right) = \Phi(z) - \Phi(y),
\]

\[
P\left( \{ X > z \} \right) = 1 - P\left( \{ X \leq z \} \right) = 1 - \Phi(z).
\]
The values of $\Phi(z)$ can be found also in the prevalent table calculations for the computer under the name „Distribution function of the standard normal distribution“. (→ Ex. 5.3)

The average value or the "normal value" in our examples, around which all other values are evenly distributed, is the parameter $\mu$, which is also the expected value of the normal distribution. If the random variable $X$ is not distributed according to the standard normal distribution, but only normally distributed, nevertheless the table for the standard normal distribution can still be used by employing a simple transformation.

**Reduction of the standard normal distribution:** If the random variable $X$ is normally distributed, the random variable $Z = \frac{X - \mu}{\sigma}$ is distributed according to standard normal distribution.

It is thus true:

$$P\left(\{X \leq y\}\right) = P\left(\left\{Z \leq \frac{y - \mu}{\sigma}\right\}\right) = \Phi\left(\left\{\frac{y - \mu}{\sigma}\right\}\right).$$

A critical point in the application of normal distribution on modelling of length, weight, etc. is that a normally distributed random variable with positive probability can also take on the negative values. This probability is however often small to the point of disappearing, due to the fact that density of the normal distribution decreases very intensely if one gains distance from expected value $\mu$. In such a way it is true e.g.

$$P\left(\{\mu - 2 \cdot \sigma \leq X \leq \mu + 2 \cdot \sigma\}\right) = \Phi\left(\frac{\mu + 2 \cdot \sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 2 \cdot \sigma - \mu}{\sigma}\right) = 2 \cdot \Phi(2) - 1 = 0.9544,$$

$$P\left(\{\mu - 3 \cdot \sigma \leq X \leq \mu + 3 \cdot \sigma\}\right) = \Phi\left(\frac{\mu + 3 \cdot \sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3 \cdot \sigma - \mu}{\sigma}\right) = 2 \cdot \Phi(3) - 1 = 0.9974.$$

The values outside the interval $[\mu - 3 \cdot \sigma, \mu + 3 \cdot \sigma]$ consequently appear with a probability of approximately 0.26 % at most. This practically means that we will hardly ever observe such values.

As a calculation example we will observe the workpiece which is measured by different people. We will assume that the workpiece was measured accurately in the middle and let the standard deviation of the measuring error be exactly 10 mm. The measuring error is modelled as a normally distributed random variable $X$ with expected value $\mu = 0$ and standard deviation $\sigma = 10$. How big is then the probability of making a measuring error of less than 5 mm? Firstly one can convert the random variable into standard normal distribution

$$P\left(\{\mu - 2 \cdot \sigma \leq X \leq \mu + 2 \cdot \sigma\}\right) = P\left(\left\{-5 \leq X - 0 \leq 5\right\}\right) = P\left(\left\{-\frac{5}{10} \leq Z \leq \frac{5}{10}\right\}\right) = P\left(\left\{-\frac{1}{2} < Z \leq \frac{1}{2}\right\}\right),$$

and additionally one can read off the values of the standard normal distribution from the table

$$P\left(\left\{-\frac{1}{2} < Z \leq \frac{1}{2}\right\}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right) = 2 \cdot \Phi\left(\frac{1}{2}\right) - 1 = 0.383.$$

The probability of achieving the deviation of the measuring result of less than 5 mm is bigger than 1/3.
Let us now imagine that the 20-year-old women living in one city had an average body height of 170 cm and that the standard deviation of the body height amounted to 9 cm. \( X \) would then be a normally distributed random variable with expected value \( \mu = 170 \) and standard deviation \( \sigma = 9 \).

How big would the probability of a randomly selected woman being bigger than 190 cm be?

\[
P\left(\{X > 190\}\right) = P\left(\left\{ \frac{X - 170}{9} > \frac{190 - 170}{9} \right\}\right) = P\left(\left\{ Z > \frac{20}{9} \right\}\right).
\]

\[
P\left(\left\{ Z > \frac{20}{9} \right\}\right) = 1 - \Phi\left(\frac{20}{9}\right) = 0.0132.
\]

The probability would be significantly smaller than 2% (although in our example-city there are seemingly many big women).

**Exercises**

**Ex.5.1** Are the following experiments Bernoulli experiments?

a) Tossing a dice

b) Throwing a dice and checking whether the number is even or odd

c) Number of parents (per child), which are present at the parents' meeting

d) The „loves me – loves me not - game“ with a flower

**Ex.5.2** The probability of a bag of gummy bears containing a number of bears which can be divided by three is 1/3. How big is the probability of three children who want to share fairly fighting over the remaining bears? Observe the random variable \( X \) which takes on the value of 1 if the children fight, and otherwise the value of 0! Calculate the expected value and variance of \( X \! \)

**Ex.5.3** Calculate the probability for the following binomially distributed random variables:

a) In a family with five children and no twins how big is the probability of only having boys? (Choose the above probability!)

b) In case of a two-time throw of a coin, how big is the probability of throwing heads exactly once?

c) Mrs Schmitt buys mineral water in a hurry. Due to the fact that she wants to buy carbonated and non-carbonated sparkling mineral water, she mixes a case of 12 bottles. Because she is in a hurry, she takes the bottles randomly from the shelf. How big is the probability of her getting exactly the equal number of bottles of both kinds?

d) How big is the probability that Mrs Schmitt has no non-carbonated water in her randomly assembled mineral water case?

e) How big is the probability of the following streak of bad luck during the „Mensch-Ärger-Dich-Nicht“ game: a three-time dice throw and no six?

f) The probability of buying a faulty light bulb is 1/100. How big is the probability that, when buying a four-pack on sale, there is no faulty light bulb in it?

**Ex.5.4** Calculate the probability for the following binomially distributed random variables:

a) An ornithologist is observing birds in the park. The probability of the spotted bird being a sparrow is 80%. If now the bird scientist saw 15 individual (why is this important?) birds, how big is the probability that at least four birds would not be sparrows?

b) A service technician of a computer company fixes damaged computers in 90% of all cases in exactly 15 minutes and in all other cases it takes him 40 minutes. One morning he gets 14 assignments at eight o’clock. How big is the probability of him not making it to his lunch break at exactly 12 o’clock?
Ex.5.5 Explain in detail why it is true for the random variable $X$, distributed according to the standard normal distribution, that $P(y < X \leq z) = \Phi(z) - \Phi(y)$ and $P(X > z) = 1 - P(X \leq z) = 1 - \Phi(z)$!

(see also Chapter 4)

Ex.5.6 For the following problems you will need a table of the standard normal distribution. If you have access to a computer with appropriate software, try to make a clearly laid out table!

Transfer the values from the examples in the above text (page 150)!

a) How big is the probability of making a measuring error of less than 7 mm while measuring the workpiece?

b) How big is the probability of making a measuring error of more than 8 mm?

c) It turns out that one person measured the workpiece 2 cm too long. How big is the probability of something like that happening?

d) In the town described in the text how big is the probability of one accidentally running into a 20-year-old woman who is smaller than 152 cm?

e) How big is the probability that the accidentally spotted 20-year-old woman is actually circa 170 cm tall, if by "circa 170" we mean all women between 168 and 172 cm?

Ex.5.7 A doctor notices that the length of his patient talks is approximately normally distributed and in fact with expected value of 12 minutes and standard deviation of 3 minutes.

a) How big is the probability of one randomly selected talk being shorter than 10 minutes?

b) The pharmaceuticals sales representative knows that he can talk to the doctor after the next patient. How big is the probability of him having to wait longer than 20 minutes?

c) How big is the probability that a randomly selected talk is longer than 30 minutes? Make a judgement, based on this result, on whether adopting normal distribution is appropriate for the duration of the patient talks!

Ex.5.8 After setting up a roofed terrace the beer garden owner Fredel thinks that the number of his guests per day in the summertime is approximately normally distributed. He enters his daily observations into his new computer program and after many clicks and calculations he is of the opinion that his data are normally distributed with an expected value of 200 guests per day and a standard deviation of 50.

a) In horror he notices that his beer is almost out and that the beer for today will in principle suffice for only 210 guests. How big is the probability of him having disappointed guests on this particular day?

b) The innkeeper believes that the atmosphere in his garden is the best when there are roughly 170 to 240 guests. What is the probability of that optimal number of guests appearing on a randomly selected day?

c) A mathematician who gladly frequents that beer garden, and after a few beers starts a conversation with the owner, believes that Fredel applied his computer program a bit sloppily. Firstly in case of a normal distribution all real values are considered a possible result and not only the natural numbers. Secondly the binomial distribution would be a much better choice for the number of his guests per day. The mathematician suggests that in a city of 300 000 inhabitants each person decides with the same probability $p$ whether they want to come to Fredel's beer garden today or not. $p$ would then have to be appropriately determined, so that the expected value of the binomial distribution is at 200 exactly.
Determine an appropriate p! Additionally calculate the standard deviation and compare this to the normal distribution model! Think of advantages and disadvantages of accepting binomial distribution as well as normal distribution for modelling the number of guests per day!

5.4 Continuation of discussion: Arbitrage – A lot of money out of nowhere

Who would have thought otherwise, it was Selina who did not want to eat her croissant and decided that the explanations should start with a simpler model.

Sebastian: The one-period binomial model is the simplest model for a stock price that you can imagine. Here, I'll give you an example. We will assume that the price of a stock develops according to the following diagram:

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>120</td>
</tr>
</tbody>
</table>
```

i.e. the stock price of 100 today can either increase to 120 after a year or decrease to 90.

Nadine: Now, this has, however, absolutely nothing to do with reality!

Oliver: Oh, yes it does! We can see that the price can rise or drop, and randomly at that. It increases with the probability of $p$ and decreases with the probability of $1 - p$. Because here we are dealing in principle with a binomial distribution and we are looking at only one period, this model is also called the one-period binomial model.

Sebastian: And now, Selina, imagine that within this simple model the stock price never drops, but in the worst of cases would rise to only 110 and the current market interest for risk-free invested money, and loans as well, would be smaller than 10 %.

Selina: I see an opportunity here to become stinking rich. I simply borrow enough money at the market interest rate. I use that to buy as many stocks as possible at 100 and sell them after a year for at least 110. Additionally I pay off the loan, including the interest, per borrowed 100 that is then actually less than 110 because Sebastian set the market interest to less than 10 %. Per every purchased stock I make certain profit and after a year I'm a millionaire.

Sebastian: I thought that's exactly what you'd immediately notice. This is by the way called arbitrage opportunity, in other words and opportunity to make profit without one's own capital and risk. In our models I want to from now on set it straight that there is no arbitrage opportunity.

Selina: Why on earth would you do that?

Sebastian: Let's assume that there is this arbitrage opportunity. Then there are also many Selina's in the world who immediately notice that. They all jump at the stock and want to buy it. Based on big demand the price of the stock would in a second rise to a level where there is no arbitrage opportunity.
Oliver: By the way, there is another possibility of developing arbitrage. Imagine if in a simple model the stock price went only down.

Selina: Then I'd borrow the stock somewhere and afterwards sell it. Borrowing stocks and subsequently selling them is incidentally called short-selling. However it's legally very severely limited. I would invest the money obtained in such a way as fixed deposit, after a year take interest, then buy the replacement stock cheaply on the market and give it back. Even in case of an interest rate of only 1 % per year, I would have made profit per stock.

Sebastian: Exactly. And if many people start acting like you, due to the fact that all of a sudden many would want to sell this stock, the price would drop so dramatically that you could kiss this arbitrage opportunity goodbye.

Selina: Pity...

The risk-free profit that piled up before Selina's inner eye now crashes with a big rumble. It is sensible to cancel arbitrage opportunities when considering a stock price model. This is why in the following we want to concentrate a bit more on the arbitrage opportunities.

Discussion 2:

It is recommended to discuss the concept of arbitrage opportunity in some detail. Some possible aspects could be:
- Is an investment in a risk-free bond an arbitrage opportunity?
- Is free lottery participation an arbitrage opportunity?
- How can one transfer the concept of arbitrage opportunity to other areas of life?
- Do you believe that there are arbitrage opportunities (on the market, in life, etc.)?

5.5 Background: Arbitrage in a one-period binomial model

Arbitrage

Firstly we want to pose an informal definition:

An arbitrage opportunity is the opportunity to gain profit without one's own capital whereby at the same time there is no risk of suffering losses.

This will now be algebraically specified:

**Definition:**

Let $X(t)$ be the capital of an investor who is investing on the stock market, in which case $t$ filters all the time periods between 0 ("today") and the time horizon $T$. We can then say that there is an arbitrage opportunity for the investor if it is possible that he starts with the capital $X(0)=0$ and it is true for his closing capital $X(T)$

$$X(T) \geq 0 \quad \text{and} \quad P\left\{X(T) > 0\right\} > 0,$$

i.e. in the end no debts ever arise for the investor. However the prospects of gaining a strictly positive closing capital have a positive probability.

Although the above definition is true for general securities, we want to firstly restrict ourselves to affiliating conditions, so that in a one-period binomial model there is no arbitrage opportunity. For this purpose we want to firstly provide a formal description of the stock market characterized by means of the one-period binomial model:
The stock market in a one-period binomial model:

We will assume that on our market at the time point $t=0$ there are both of the following investment opportunities:

- Purchase and (short-) selling of stocks with today’s price $P_1(0) = p_1 > 0$ and future price $P_1(T) = p_1(1+rT)$ with probability $p$ and $P_1(T) = p_1(1+rT)(1-p)$ with probability $(1-p)$, whereby $u > d$ is true.

- Fixed deposit or loan at an interest rate of $r \geq 0$, in which case we adopt the continuous return in a time frame $[0,T]$, i.e. the capital development of a monetary unit is given by means of

$$P_0(0) = 1, \quad P_0(T) = e^{rT}$$

Note: The continuous return is chosen here in regard to the later introduced Black-Scholes model (see Section 5.6/7/8). If we discuss only time discrete models, to simplify matters we can also adopt a singular return at $[0,T]$, thus

$$P_0(0) = 1, \quad P_0(T) = 1 + r \cdot T,$$

If one selected $r$ instead of the rate of interest $r^* = 1/T(e^{rT} - 1)$, both return types would lead to the same value $P_0(T)$.

Consequently the investor can distribute his capital in $t=0$ and either buy or borrow stocks, as well as invest or lend money. If he wants to e.g. buy more stocks than his opening capital of $x$ permits, he has to take out an appropriate loan. If he invests less than $x$ monetary units in the stock, in our model he has to invest the rest in a fixed deposit.

In the binomial model according to our definition the number of upward movements of the stock price is $B(1,p)$-distributed, from which the name binomial model is derived.

**Definition:**

By *trading strategy* (in the one-period binomial model) we mean a pair $(f, g)$ in IRxIR with

$$x = f + g \cdot p_1,$$

in which case $f$ describes the face amount invested in $t=0$ and $g$ represents the number of stocks kept in $t=0$.

If $f$ is a negative number, this means that a loan was taken up. If $g$ is negative, a short sale has taken place.

Due to the fact that in a one-period model one trades only at the beginning and then leaves their security combination unchanged up to the final point in time, one also calls this a buy-and-hold-strategy. This trading strategy leads to the closing capital

$$X(T) = \begin{cases} 
-f \cdot p_1 \cdot e^{rT} + g \cdot p_1 \cdot u & \text{with probability } p \\
-f \cdot p_1 \cdot e^{rT} + g \cdot p_1 \cdot d & \text{with probability } (1-p)
\end{cases}.$$
Here one can see clearly that the closing capital random variable $X(T)$, once the trading strategy has been selected, can take on only two possible values. Only if one invests their total face capital, one knows already at the point $t=0$ which capital one will have at the final point $T$.

However if one knew that even in the worst of cases the paid interest on the stock would be better (in the sense of: bigger or equal) than the one on fixed deposit (or loan), in other words if one took on $u > d \geq e^{rT}$, one would take up a loan today, buy stocks with it, then pay off the loan in $T$ and take the remaining stock gain. Formally one would thus choose $-f = g \cdot p_1 > 0$ and then get

$$x = X(0) = 0$$

$$X(T) = \begin{cases} -g \cdot p_1 \cdot e^{rT} + g \cdot p_1 \cdot u & \text{with probability } p \\ -g \cdot p_1 \cdot e^{rT} + g \cdot p_1 \cdot d & \text{with probability } (1-p) \end{cases}$$

In both cases the closing capital $X(T)$ is, due to the acquisition $u > d \geq e^{rT}$, non-negative and in the first case even strictly positive. One could consequently gain arbitrage profit. Analogously there arises an arbitrage opportunity if the stock developed from bad to worse compared to the risk-free financial investment. In order to avoid such arbitrage opportunities, we will thus

$$d < e^{rT} < u \quad \text{„No-Arbitrage Constraint“}$$

in a one-period binomial model. Implicitly we also claim $0 < p < 1$.

**Exercises**

**Ex.5.9** Describe formally and in detail, in your own words, the arbitrage opportunity in a one-period binomial model, which would be generated if the stock developed in the best of cases (in the sense of: smaller or equal) worse than the fixed deposit!

**Ex.5.10** Calculate according to the given trading strategy $(f, g)$ and the provided opening capital $x > 0$:

a) $E(X(T))$

b) $Var(X(T))$

**Ex.5.11** The following one-period binomial model is given by means of $r = 0.05$, $u = 1.2$, $d = 1$, $T = 1$, $p_1 = 100$, $p = 0.75$ (designation same as above). Imagine being in possession of an opening capital of 1000 €.

a) Determine all trading strategies $(f, g)$ with $E(X(T)) \geq 1100$. Which of these has the minimal variance?

b) Is it possible to indicate a trading strategy with $E(X(T)) = 1000$? Support and describe this in detail!

**Ex.5.12** Is the following "binomial model with two stocks" free of arbitrage? Produce a drawing!

In our market model we have indicated a face security with a continuous return of 0.01. Aside from that there are two stocks, both at a starting price 100. After a year the value of the first stock changes with the probability $p$ to 120 and with the probability $(1-p)$ to 80. After a year the value of the second stock changes with the probability $p$ to 115, and to 90 with the probability $(1-p)$. Both stocks are however not independent of one another, meaning if the price of one stock decreases, so does the price of the other; if the price of one increases, so does the price of the other. Yet there should be a possibility of buying them completely independently of each other.

**Ex.5.13** A small remark on the everyday economic life: In reality there are in fact sometimes arbitrage opportunities. However there are plenty of people (not only dealers!), who purposefully look for these arbitrage opportunities – the so-called arbitrageur – which is why these opportunities never persist very long and the chances of profit are mostly small. Reflect on the current ex-
amples from everyday life which so-to-say seem to represent "arbitrage opportunities" and discuss about them (e.g. the opening of a new store just around the corner with free coffee and cookies).

5.6 Continuation of discussion: More reality – the multi-period binomial model

Yeah, yeah, though Selina has not taken a bite off her croissant, she’ll have plenty of time to chew on the insight that reasonable stock models are arbitrage-free. After Oliver mentions that one also often calls arbitrage „free lunch“, her empty stomach starts grumbling at the mere thought of it. According to the motto „A fat belly, a lean brain“ and with the image of herself being able to buy a size 36 elegant this evening, she lunges over-motivated at the investigation of new mathematical domains.

Selina: What’s the situation with the binomial model if there are more stocks?

Sebastian: That’s more difficult. But it is entirely not problematic to expand the model to more periods. One impends simply many models one after the other. The result of this is a big branched tree with many branches and we speak of the multi-period binomial model. It is wonderfully suited for simulating stock price development.

Chart 5.5 Binomial tree

Nadine: Yeah, but Sebastian, you surely aren’t trying to explain to us that this has something to do with reality. At the end of a 4-period binomial model there are only four possible stock prices! How many periods do we really need here for some really realistic modelling?

Sebastian: 1000.

Selina: Come again? You can’t be serious!

Sebastian: Sure I am. Yeah, ok, 1000 is of course no unalterable number. What I mean is only that one should select that time between two points, in other words the length of the period, as very small in order to have many possible prices at your disposal at the final point of the observation period.

Oliver: Ah, exactly! A lot of small ups and downs result in a shape which looks like a real stock price.
Selina: Such an irregular zigzagged up and down? Like for example this stock price of Gabriel Müll Inc. here in my business paper?

![Stock Price Chart](image)

**Chart 5.6 Fictional stock price of Gabriel Müll Inc.**

I simply can't believe this. Such a binomial tree looks very regular, how can such a chaotic stock price development come out of that?

Nadine: You just have to also pay attention that the tree contains all possible price developments. In fact you only see one single consequence of ups and downs. This looks pretty zigzagged. Oliver, couldn't you swiftly simulate something on the laptop?

Oliver: Already thought something like that would come up. But of course, I'd be happy to. I will simply choose $u=1.013$ and $d=0.99$. Let the opening price of the stock be 100.

Nadine: How did you come up with these numbers?

Oliver: 13 is my lucky number.

Selina: Now then, no need to wonder when you get struck by bad luck!

Oliver: Now I still have to gamble in each point in time on whether the current stock price is multiplied by $u$ or by $d$.

Sebastian: Gambling isn't gonna help you much there because a $u$ appears with the success probability of $p$ and $d$ with the probability $(1-p)$.

Nadine: Don't be such a know-it-all. Oliver is surely using a random number generator.

Oliver: That's right! And here is also my simulation.
Looks good, doesn’t it? By the way, I chose $p=1/2$, one could have thrown a dice on that one in case of an emergency.

Selina: Looks totally real!

Nadine: I’m not that happy with it, you should have worked with more periods. If you take a good look at it, you can actually see a certain regularity.

Oliver: But I chose $u$ and $d$ marvellously, right? $u$ should not be bigger than 1 because otherwise the price could get gigantic. Likewise $d$ should not be smaller than 1 because otherwise it hits 0 fast.

Sebastian: Can you still remember the arbitrage considerations in a one-period case? We need

$$d < e^{rT} < u.$$  

Due to the fact that here in a multi-period binomial model we divide time in many very tiny pieces, our $T$ is very small and almost equal to zero. This then also means that $e^{rT} \approx 1$. Oliver’s choice thus ensures that this 100-period binomial model is arbitrage-free.

Nadine: That’s all good! But we then have as closing price

$$P_f(1) = 100 \cdot 1.013^{X} \cdot 0.99^{100-X},$$

in which the random variable $X$ is binomially distributed with $X - B(100, 1/2)$. With such a binomial distribution we can do only elaborate calculations. Consider all the binomial coefficients alone!

Sebastian: Can it be that you’re about to introduce the normal distribution and the Black-Scholes model.

Nadine: Exactly.

Before we continue listening in on the conversation, we firstly want to take care of some mathematical details and observe the basic principles of the $n$-period binomial model, the Black-Scholes model, and their relationships.

### 5.7 Basic principles of mathematics: The $n$-period binomial model and the Black-Scholes model

#### The multi-period binomial model

The multi-period binomial model represents the direct generalization of a one-period binomial model from Section 5.5. In literature it is also known as the Cox-Ross-Rubinstein model and will
be introduced in the following. On the one hand one can understand it as a nice, simple model by means of which one can exemplify many basic principles of financial mathematics. However, it can also be perceived as an approximation for complex models such as the famous Black-Scholes model.

We will now observe the development of a stock price $P_1^{(n)}(T)$ in an $n$-period binomial model with the time horizon $T$. In an $n$-period binomial model price changes (and trade) at times $j \cdot T/n$ with $j = 1, \ldots, n$ take place, respectively. The development of a stock price in the Cox-Ross-Rubinstein model is reproduced by means of the following diagram in which we restrict ourselves, for the sake of simpler presentability, to the case $n=2$:

![Diagram of a two-period binomial model]

**Chart 5.8** Stock price development in a two-period binomial model

The stock price thus acts like a tree which is composed of singular one-period binomial models (also called **tree models**). The addition factors $u$ and $d$ are, much like the probability $p$, equal for a price increase in each knot, so that the price of the stock at the particular time $j \cdot T/n$ is distinctly determined by the number of the previously occurred upward movements of the stock price. The name **binomial model** is explained by the fact that the number $X_n$ of the upward movements in an $n$-period binomial model suffices for a binomial distribution with parameters $n$ and $p$. This is due to the fact that $X$ is the sum of $n$ independent zero-one-variables $X_i$ which each take on the value of one, if at time $i \cdot T/n$ a price increase takes place:

$$X_n \sim B(n, p).$$

The stock price results then in

$$P_1^{(n)}(T) = p_1 \cdot u^{X_n} \cdot d^{n-X_n} = p_1 \cdot \exp\left( X_n \cdot \ln\left(\frac{u}{d}\right) + n \cdot \ln(d) \right).$$

From this example one can also see that the stock price in the $n$-period binomial model in the final point in time $T$ can adopt exactly $n+1$ different values.

Much like in a one-period binomial model here we also assume that for each period there is an opportunity of investing or receiving money at a risk-free, continuous rate of interest $r \geq 0$. A monetary unit which is invested risk-free at the time $t = 0$ thus develops as follows:

$$P_0(t) = e^{rt}, \quad t = 0, \frac{T}{n}, 2 \cdot \frac{T}{n}, \ldots, T.$$ 

It is easy to check whether a market model generated in such manner is arbitrage-free if and only if the relationship

$$d < e^{rT/n} < u$$

is true. One can also easily re-examine that

$$E\left(P_1(T)\right) = p_1 \cdot \left( p \cdot u + (1-p) \cdot d \right)^n.$$

(→Ex.5.14)
**n-period binomial model:**

\[ P_0(t) = e^{rt} \]

\[ P_1(t) = p_1 \cdot u^{k_1} \cdot d^{k-k_1}, \quad t = k \cdot \frac{T}{n}, \quad k \in \{0,1,...,n\} \]

\[ X_k \sim B(k, p) \]

**The de Moivre-Laplace theorem**

If one now increases the number \( n \) in the \( n \)-period binomial model, i.e. if one opted for an ever more refined timing, we might pose a question whether for big \( n \) the result is something like a marginal distribution. The answer to this question is yes and emphasizes the great importance of normal distribution. It is based on the **the de Moivre-Laplace theorem**. This theorem proves that for big values of \( n \) the binomial distribution \( B(n,p) \) with parameters \( n \) and \( p \) (in other words the distribution of the number of successes in case of \( n \) 0-1-experiments, which were conducted independently of one another and in which case the success probability amounts to \( p \), respectively) can be approximated by means of normal distribution (more specifically: the normal distribution with the same expected value \( np \) and the same variance \( np(1-p) \)).

**The de Moivre-Laplace theorem:**

If \( X \) exhibits a \( B(n, p) \)-distribution, then

\[
\frac{X - E(X)}{\sqrt{Var(X)}} = \frac{X - np}{\sqrt{np(1-p)}}
\]

is approximately distributed according to standard normal distribution in case of big \( n \), i.e. it is true for big \( n \) (rule of thumb: \( n \cdot p \cdot (1-p) \geq 9 \)):

\[
P\left( \frac{X - np}{\sqrt{np(1-p)}} \leq x \right) = \Phi(x),
\]

in which case \( \Phi(x) \) is the distribution function of the standard normal distribution.

This convergence relationship can be for example also visually demonstrated by means of the **Quincunx**. In Chart 5.9 we will clarify this by comparing the probability function (presented in form of a histogram) of the \( B(20, 0.5) \)-distribution with the density of the \( N(10, 5) \)-distribution, a normal distribution with expected value 10 and variance 5. The deviations are therefore very low already for small \( n \).
The big advantage of this approximation consists in a simple possibility of calculating probability for binomially distributed random variables. It is therefore true for the binomially distributed random variable $X$

$$P\left(\{X \leq k\}\right) = \sum_{j=0}^{k} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k},$$

which by virtue of factorials for big $k$ requires complex calculations. By means of the theorem by de Moivre-Laplace, one obtains for big $n$ approximately

$$P\left(\{X \leq k\}\right) = P\left(\left\{ \frac{X - np}{\sqrt{np(1-p)}} \leq \frac{k - np}{\sqrt{np(1-p)}} \right\}\right) = \Phi\left( \frac{k - np}{\sqrt{np(1-p)}} \right).$$

The value of the standard normal distribution can be now simply read off in the table.  
(→Ex.5.15, Ex.5.16)

**The Black-Scholes model**

The role of the normal distribution as a marginal distribution of the binomial distribution offers us the opportunity to interpret the so-called Black-Scholes model as a marginal model of a sequence of ever more refined binomial models.

For this purpose we will firstly compose the basic principles of the Black-Scholes model. Just as in the binomial model, in this one there is a risk-free financial investment in case of which we adopt a continuous return at the rate of interest $r$. Consequently, for the development $P_0(t)$ of a monetary unit, which is invested without risk at time $t = 0$, we get

$$P_0(t) = e^{rt}, \quad t \in [0, T].$$

The temporal development of the stock price $P_1(t)$ is modelled according to

$$P_1(t) = P_0(t) \cdot e^{\left(\frac{b - \frac{1}{2} \sigma^2}{\sqrt{T}}\right) + \sigma W(t)}, \quad t \in [0, T]$$

in which case $r$, $b$ and $\sigma$ are fixed real numbers, the meaning of which will be deduced later. The most important component of the Black-Scholes model is the random variable $W(t)$, more specifically: the set of random variables $\{W(t), t \in [0,T]\}$. Such a set in which the index is a time variable and, as a result the development of a random experiment, is described over a space of time, is called a stochastic process. The random variable $W(t)$ depicts the Brownian movement,
which we will go into subsequently. The basic feature for understanding $P_1(t)$ is that $W(t)\sim N(0,t)$ is true. We call the stochastic process $P_1(t)$ also geometric Brownian movement. (→Ex.5.17)

The Brownian movement

We will now busy ourselves with the stochastic process $\{W(t), t\in [0,T]\}$, which is designated as the Brownian movement or also the Wiener process. It is determined by the fact that $W(t)$ is a normally distributed random variable with expected value zero and variance $t$, therefore it is true that

$$W(t) \sim N(0,t).$$

Additionally $W(t)$ as a function of $t$ (thus as a stochastic process) should be a continuous function and satisfy demands

i) $W(0) = 0$

ii) $W(t) - W(s) \sim N(0,t-s)$ for $t > s$ „normally distributed accretion“

iii) $W(t) - W(s)$ is independent of $W(r) - W(u)$ for $t > s \geq r > u$ „independent accretion“

To illustrate this in Chart 5.10, we will present a simulated path of the Brownian movement, and in doing so the issue is the possible result of the belonging random experiment. In Section 5.8 we will explain how to create such simulations.

![Chart 5.10 Simulated path of the Brownian movement $W(t)$](image)

Immediately we are struck by a very irregular, zigzagged course of $W(t)$ (one ought to actually write $W(t, \omega)$ because for each $\omega \in \Omega$ we get a different course of the function $W(t)$, however for the sake of clarity we will here dispense with the explicit indication of dependency of $\omega$). We can show effectively that $W(t)$ as a function of $t$ is in no $t \in [0,T]$ differentiable! This at first seemingly very strange characteristic is for stock price models imperatively necessary. This is due to the fact that one could e.g. from a positive derivative in $t$ immediately conclude that the stock price would at any moment definitely increase. This would naturally be an escapist notion. (→Ex.5.18)

Characteristics of the stock price in the Black-Scholes model

From the characteristics of the Brownian movement (for this section we only need the feature $W(t)\sim N(0,t)$) one can conclude that for the stock price $P_1(t)$ it is true that

$$E(P_1(t)) = p_t \cdot e^{bt},$$
The middle stock price $E(P_1(t))$ acts as a fixed deposit account on which return is continuously paid at the interest rate $b$, of the middle (time continuous) stock yield per time unit. The value $b$ is designated as the **medial rate of return** of $P_1(t)$. The standard deviation of the time continuous stock yield per time unit, $\sigma$, is named **volatility** of the stock. It is "the" measure for the fluctuation margin of the stock price. Its importance will become even clearer in the chapter "Option pricing".

**Is the Black-Scholes model arbitrage-free?**

It can be in fact shown that the Black-Scholes model is arbitrage-free. However, to do this we would require technical aids which we cannot introduce here (see e.g. Korn and Korn (2001)). A heuristic justification for the no-arbitrage condition is e.g. that for all time points $t,s$ with $t>s$ it is true that

$$\ln\left(\frac{P_1(t)}{P_1(s)}\right) - N\left((b - \frac{1}{2} \sigma^2) \cdot (t-s), \sigma^2 \cdot (t-s)\right).$$

In this way it is ensured that the time continuous stock yield with positive probability is bigger, as well as smaller than the time continuous fixed deposit yield of $r \cdot (t-s)$, because the normal distribution exceeds arbitrarily large and arbitrarily small values with positive probability.

**How are the binomial and Black-Scholes model connected?**

In order for the stock price $P_1^{(n)}(T)$ in a binomial model to be able to converge against the stock price $P_1(T)$ in the Black-Scholes model, for the increasing number of periods $n$, at least two conditions have to be fulfilled:

- In a binomial model the time period between two trading periods $\Delta t = T/n$ has to hit zero, so that the continuous model with constant trading opportunities can appear as a marginal case.
- At the same time the "addition factors" $u$ and $d$ converge against one, so that the resulting marginal process can be a continuous process (as function of time).

For $u$ and $d$ we will apply the approach

$$u = u(\Delta t) = \exp\left(\beta \cdot \Delta t + \sigma \frac{1-p}{\sqrt{p(1-p)} \sqrt{\Delta t}}\right), \quad d = d(\Delta t) = \exp\left(\beta \cdot \Delta t - \sigma \frac{p}{\sqrt{p(1-p)} \sqrt{\Delta t}}\right),$$

in which case $\beta$ and $\sigma$ (with $\sigma > 0$) are given real numbers (in this case the results of the above equations are $u$ and $d$) as well as $p \in (0,1)$. We will further assume that $\Delta t$ is already so small that $u > 1 > d$ is true. In that case $\beta$ and $\sigma$ are presented in the following way

$$\beta = \frac{\ln(d) + p(\ln(u) - \ln(d))}{\sqrt{\Delta t}}, \quad \sigma = \frac{\ln(u) - \ln(d)}{\sqrt{\Delta t} \sqrt{p(1-p)}}.$$

We now have the sequences of values of $u$ and $d$ available, which each converge against one in favour of increasing $n$ (and in fact monotonely from above or below). By means of the above depcition of $u$ and $d$ one obtains

$$P_1^{(n)}(T) = p_1 \cdot \exp\left(X_n \cdot \ln\left(\frac{u}{d}\right) + n \cdot \ln(d)\right) = p_1 \cdot \exp\left(\frac{X_n - np}{np(1-p)} \sigma \sqrt{T} + \beta \cdot T\right).$$

According to the de Moivre-Laplace theorem, it follows that in the above equation the distribution of the exponents on the right side is asymptotically (i.e. for $n \to \infty$) equal to that of the exponents of
if one sets $b=\beta + \frac{1}{2}\sigma^2$. All in all we obtain the convergence of the binomial model at time $T$ against the stock price in the Black-Scholes model at the same point in time. The common convergence in all points $t\in[0, T]$ of the stock price, made continuous from the binomial model by means of linear interpolation, against the Black-Scholes model can only be demonstrated by means of penetrative mathematical propositions (see e.g. Korn and Korn (2001)).

**Note**

In order to model more than one stock price at the same time, in the binomial, as well as the Black-Scholes model, we need multidimensional random components. These will not be discussed here due to their complexity.

**Exercises**

**Ex.5.14** Observe the 8-period binomial model with the parameters $u=1.1, d=1.5, T=1, p=0.4$ and with the risk-free continuous return of $r=1.15$.

a) Is the model arbitrage-free?
b) Calculate all possible stock prices in this model at the point in time $T=1$!
c) Calculate the price development of the stock if the random variable $X_n$ of the upward movement consecutively takes on the values 0, 1, 0, 0, 1, 1, 0, 0! Indicate two more possibilities for the development of the random variable $X_n$, with which the stock has the equal price at the final point in time!

**Ex.5.15** In a very opinionated village with 1300 elective inhabitants each person decides rather accidentally whether they will take part in the local elections today or not, and in fact the probability of voting today amounts to $p=2/5$. Calculate the probability of a voter turnout in this village of

a) over 50 %!
b) below 80 %!

**Ex.5.16** In a big school with 1800 students the probability that a randomly selected student has at least one F in their final report amounts to approximately 1/20.

a) Calculate the probability that in a class of 30 students more than 5 students have at least one F in the report!
b) Calculate the probability that in the whole grade with 200 students more than 20 students have at least one F in their report!

**Ex.5.17** We will observe the stock price in the Black-Scholes model with parameters $p_1=100, b=0.1, \sigma=0.3, T=1$.

a) Calculate $P\left(\{P_1(T) \in [90,110]\}\right)$!
b) Indicate the boundaries of the stock price by means of the features of normal distribution, so that it is true that

$$P\left(\{P_1(T) \in [a_1,a_2]\}\right) \geq 0.95.$$

c)* Specify a general solution for optional parameters!

**Ex.5.18** We will observe the following Brownian movement:

$W(0)=0, W(0.2)=0.1, W(0.4)=0.05, W(0.6)=-0.1, W(0.8)=-0.15, W(1)=-0.1$.

a) Sketch this Brownian movement!
b) Indicate the probability that the actual value $\pm 0.05$ can be observed! (In doing this always indicate which distribution is observed at the moment!)
5.8 Continuation of discussion: Totally cool – a continuous stock price model

Unhappy about, in her opinion, the escapist binomial model, Nadine dug out the chocolate from hand bag for comfort and during the last talk already devoured half a tablet. Since she can now finally strike with her explanations, she packs the rest in aluminum foil, puts it on her empty plate and takes a big black pen from her bag.

Nadine: Take a look at the annual stock chart of Gabriel Müll Inc. once again (see also Chart 5.6 and 5.11)! Here everything seems very angular and wild, as if one accidentally scribbled up and down with a pen. And now I will plot a trend line with this pen.

Selina: What’s that supposed to be? You’re simply scribbling with a marker all over my business paper!

Nadine: No worries, you can erase it again. Now, it looks as if the stock price were composed of two components. Apparently there is a long term definitive trend and short term influences, which lead to locally strongly fluctuating stock prices. As a long term trend I plotted a line which corresponds to the face capital with continuous return of ca. 20 %.

Oliver: Do I hear return of 20 %? Amazing how much money you can earn from garbage!

Nadine: But, as you can see according to the spikes, the profit fluctuates! And in fact quite irregularly and unexpectedly.

Oliver: This spiky pattern reminds me of the crazy movements of the small meager water beetles at the university pond.

Nadine: You’re quite right there. These spikes are explained by means of a specific type of the so-called Brownian movement, much like the movements of the water beetles on the pond. The term and the theory of the Brownian movement was introduced initially in order to model the movements of little particles on the water surface. And though the movement looks so abstruse, one can algebraically formulate it in a very simple manner. Isn’t that great?

Sebastian: A stochastic process with continuous paths and static and independent accretion, that’s not very simple.
Nadine: So, Sebastian, this is something meant for theoreticians. Selina, you don't have to understand this mathematical trivia. We want to only apply the market model, which I will introduce in a second, and for that one doesn't have to have the complete theoretical derivation ready. Primarily the model has to appropriately reflect the stock price performance of the realistic price development, and it does just that. In practice it is by the way known as the **Black-Scholes model** and is applied million-fold.

Sebastian: Sure. But you have to tell something concrete at some point. After all we are also mathematicians.

Nadine: Exactly! So, Selina, imagine a water beetle which is irregularly moving across the water a little bit to the right, then to the left etc. We will hold constant its every whereabouts $W(t)$ at time $t$ as the function of the time $t$ in a chart. In order for this to be a two-dimensional drawing, we will only pay attention to the deviation from the imaginary axis of the pond. Because the beetle continually moves, $W(t)$ is always a continuous function and we can therefore draw without depositing. However, we don't know the future value $W(t)$, it is a random variable. We have to thus wait the whole time to see where the beetle will jump at a particular moment and we can't draw a line in advance. If our water beetle moves according to the Brownian movement, then its currently unknown whereabouts at the time $t$ are normally distributed with variance $t$

$$W(t) \sim N(0,t).$$

Selina: So, so. Oliver, you always have something demonstrative for every occasion! Can you show us something here?

Oliver: Why, of course. I saved a simulation program on our Notebook. This program generates Brownian movements by means of a random number generator. The chart of the movements of the water beetle described by Nadine could then look similar.

![Simulation of the Brownian movement](chart5.12)

Here, Nadine, this time it's really zigzagged and irregular enough, right?

Nadine: I'm very content with it. Now we want to construct a stock price by means of this ingredient. Take a look at the chart of the Müll Inc. one more time (Chart 5.11). The stock price in the Black-Scholes model consists of two components, the trend component and the Brownian movement component:

**Stock price component 1:** - The „face trend component“, i.e. the interest on the opening price of the stock is continuously paid, see trend line.

**Stock price component 2:** The component with the **Brownian movement**, which depicts the purely accidental fluctuations of the stock price, see spikes.
It was my conscious decision not to write Component 1 + Component 2.

**Oliver:** This would namely result in sometimes negative stock prices because the normal distribution within the Brownian movement also delivers negative values. While doing this the first guy who busied himself with modelling of stock prices flunked his doctoral examination. What was his name again?

**Sebastian:** Bachelier was the name of the person who first tried to describe stock prices by means of the Brownian movement. By the way, he passed his doctoral examination, however with a relatively bad grade, which finally ruined his scientific career. It’s not so simple to convincingly model stock prices!

**Nadine:** In order to avoid negative prices, one can employ a clever trick. You can model a **logarithm of the stock price** by means of the Brownian movement and set:

Logarithm of component 1: $\ln(p_0) + \beta \cdot t$.

There is no chance here!

Logarithm of component 2: Brownian movement with volatility $\sigma > 0$, thus $\sigma W_t$.

This means that in the second component we have the time dependent, normally distributed random variable $W_t$ with $W_t \sim N(0, t)$, which is multiplied by the constant $\sigma$, or the so-called volatility.

Now we can add:

Logarithm of the stock price: $\ln(p_0) + \beta \cdot t + \sigma \cdot W_t$.

The logarithm is allowed to thereby easily take on negative values. We then obtain the price of the stock as

**Stock price at time t:** $p_0 \cdot \exp(\beta \cdot t + \sigma \cdot W_t)$,

which is, due to the exponential function, always non-negative. That’s it. I still have some chocolate left. Anybody want a piece?

**Selina:** Oh yeah, just don’t pass it all on to Oliver. Somehow I’m hungry. Now still explain to me please how the just mentioned volatility is connected to the stock figure "volatility"!

**Nadine:** The stock figure "volatility" is the standard deviation of the logarithm of the stock price. Observed in the model and based on one year, meaning $t = 1$, this is exactly volatility $\sigma$.

**Selina:** Where on earth does the value from the paper now come from?

**Sebastian:** There are market analysts which busy themselves with pricing this value, e.g. by means of this model. There is a possibility of taking as basis the pricing of the observed stock prices of the past 30 or 250 days, which then results in a 30-day or 250-day volatility. However we can also calculate the volatility from the option prices (see Chapter 7). In my opinion this results in better market estimate because these values are future-oriented. Since there are many appraisal methods, the values in different papers can be different.

**Selina:** And the value $\beta$ is the expected rate of return of my stock?

**Nadine:** Unfortunately no. Normally one writes down the model in a different way:

**Stock price at time t:** $p_0 \cdot \exp((\beta + \frac{1}{2} \cdot \sigma^2) \cdot t + \sigma \cdot W_t - \frac{1}{2} \cdot \sigma^2 \cdot t)$.

If we now substitute $\beta + \frac{1}{2} \cdot \sigma^2$ with $b$, we get:

Modelling of the stock price by means of geometric Brownian movement, the **"Black-Scholes model"**
The price of the stock $P(t)$ at time $t$ is modelled as

$$P(t) = p_0 \cdot \exp\left( \left( b - \frac{1}{2} \cdot \sigma^2 \right) \cdot t + \sigma \cdot W_t \right),$$

in which $W_t \sim N(0,t)$, $b > 0$, $\sigma > 0$.

It is true for the expected value:

$$E\{P(t)\} = p_0 \cdot e^{b \cdot t}.$$

Your desired value is then $b$.

Selina: No! This is not quite correct, the devil is in the details! You have to pay attention to whether the interest is paid continuously or annually. If we take a look at only the rates of return which apply to the annual interest payment, we have to convert the continuous interest rate $b$ into the effective interest rate. In doing so the expected rate of return of the stock per year is a little bit more than $b$!

Nadine: Oh, Selina, picky trifles are otherwise my area of expertise!

Selina: But not if we’re dealing with potential gains or losses!

Selina leans back contently in the comfortable train seat and beams at the conductor who just looked in. As she asks him to bring a few frankfurters with buns after all, everybody else joins in and in doing so orders breakfast for the third time. Sebastian’s cue that the Brownian movement can have such unexpected deflection that it can theoretically drive the stock price almost down to zero, impresses Selina only marginally.

Exercises

Ex.5.19 We will now observe a 4-period binomial model in more detail!

We will assume that the opening price of the stock is $p_1 = 50$. In each period the stock price changes at the factor $u = 1.15$ with probability $p$ or at the factor $d = 0.85$ with probability $(1 - p)$. Each time period is of the length $t = 1/4$.

a) Draw a 4-period binomial tree (on a sufficiently big piece of paper)!

b) Is this model arbitrage-free, if the interest is paid on the fixed deposit with $r = 0.05$ continuously?

c) Is this model arbitrage-free if the interest is paid continuously on the fixed deposit with $r = 0.15$?

c)* If there is additionally the possibility to invest in fixed deposit with $r > 0$ on which the interest is continuously paid, which demands do the factors $u$ and $d$ have to meet in order for the model to be arbitrage-free?

d) Simulate the possible stock price developments in this model! Additionally apply the following results of the repeated fair coin toss:

i) $0 \ 1 \ 0 \ 0$

ii) $1 \ 0 \ 0 \ 1$

„1“ means that the factor $u$ should be applied, „0“ means that the factor $d$ should be applied.

iii)* Contemplate your own simulation possibility (e.g. with the dice)!

iv) Draw a stock chart for this purpose! Does it look realistic?

e)* If the factors $u$ and $d$ changed after every period, what would change about the way the binomial tree looks?
Ex. 5.20 Let's imagine that we observed the stock over ten days and noted down the following stock prices (closing call):
25,19€, 25,90€, 26,30€, 25,00€, 24,90€, 25,25€, 25,45€, 26,20€, 26,10€, 26,70€.

a) Draw a stock chart!
b) Sketch a trend line into the stock chart!
c) Calculate the logarithm of the stock price and draw this function!
d) We will now observe the stock price development by means of the Black-Scholes model. We will assume here that \( b=0.09 \) and \( \sigma=0.3 \). Calculate now 10 different values, those which were adopted during our observations by the Brownian movement \( W \) incorporated in our model!
e) Draw this Brownian movement!
f) What is the expected rate of return of this stock according to this Black-Scholes model?

5.9 Basic principles of mathematics: Random numbers and simulations of stock prices

Simulations

Up to now in the course of this book we have on a few occasions brought up simulations, in particular stock price simulations, without getting into what a simulation is and how it can be created. A simulation is a modelled reproduction of a real event. In most of the cases one presents processes which contain one or more random components. One therefore observes a simulation in general as a random experiment in which a random variable should be created through a specified distribution. In particular one uses computer-generated simulations in order to act out the "what-happens-if scenarios". One can easily observe the results without risking economic or human losses.

Random numbers

For simulation with random components we need appropriate random variables. One can naturally create many random variables in a purely physical way, such as e.g. equipartition over the numbers 1,2,3,4,5,6, by means of tossing the dice with a fair dice or a Bernoulli distribution by means of throwing a (possibly unfairly tampered-with) coin. However this procedure is, considering the often enormously big number of belonging random numbers needed for the application, very inefficient. For this reason we will make the following assumption:

Assumption:
There is a mechanism available, by means of which one can create optionally many independent random numbers which are consistently distributed on the interval \([0,1]\).

When we say mechanism we mean, for the sake of simplicity, a function provided by the computer (typically named „random”, however it can be named otherwise depending on the type of software used). In this case we will not go into methods of number theory which are used by the computer to create (pseudo) random numbers.

After we have come into the possession of a mechanism which provides us with optionally many independent random numbers, which are equally distributed over \([0,1]\), we want to show in the following how we can by means of this mechanism create random numbers with a predetermined distribution. We will observe three different cases.

Note: Equally distributed random numbers

The equal distribution of a finite event set \( \Omega \) can be comprehended on an intuitive level. Each element of \( \Omega \) has the same probability, namely \( 1/|\Omega| \) (in which case \(|\Omega|\) represents the number of elements of \( \Omega \)). In case of the real random variable \( X \) with the equal distribution over the interval \([a,b]\) the situation is similar, each component interval of \([a,b]\) has the same probability as the
other component interval of the same length. Here the random variable \( X \) over \([a, b]\) is called **equally distributed**, if it has the following density

\[
f(x) = \begin{cases} 
\frac{1}{b-a} & \text{for } x \in [a, b] \\
0 & \text{for } x \notin [a, b]
\end{cases}
\]

**i) Discretely distributed random numbers**

We will firstly observe the case in which the random variable \( X \) should be simulated. The variable takes on only finitely many values \( x^{(1)}, \ldots, x^{(k)} \), in which case

\[
p_i = P\left\{ X = x^{(i)} \right\}, \quad i = 1, \ldots, k
\]

is true. In order to obtain a random variable with the same distribution as \( X \) from the random variable \( Y \), which has the equally distributed over \([0,1]\), we will divide the interval \([0, 1]\) in \( k \) component intervals \( I_1, \ldots, I_k \) by means of

\[
I_i = \left[ p_i + \ldots + p_{i-1}, p_i + \ldots + p_1 \right), \quad i = 1, \ldots, k - 1,
\]

\[
I_k = \left[ p_k + \ldots + p_{k-1}, 1 \right].
\]

We will then define the random variable \( Z \) in the following way:

\[
Z = x^{(i)}, \quad \text{falls } Y \in I_i.
\]

This virtually means: if the random generator (e.g. „random“) creates the number \( Y \in [0,1] \), we firstly determine the interval in which the value is located. Let us assume this to be the \( i \)-th interval in which case we set the random variable on the \( i \)-th value of the random variable \( X \). In this way the random variable \( Z \) takes on the same values as \( X \) and the distributions are concordant, i.e.

\[
P\left( X = x_i \right) = p_i = P\left( Z = x_i \right).
\]

A discrete random variable, which features a distribution in which countably many values with positive probability are assumed, can be entirely analogously simulated. One has to merely divide the interval \([0,1]\) in the same way as above into infinitely many intervals, in which case there can naturally be no final interval.

**ii) Random numbers with continuous distribution function**

If we want to create values of the random variable \( X \) which takes on uncountably many values, as for example the normal distribution or the continuous equal distribution, we need **distribution function** \( F(x) = P\left\{ X \leq x \right\} \) of the random variables. If the distribution function \( F(x) \) is continuous, we can define the generalized inverse function of \( F(x) \) as:

\[
\tilde{F}^{-1}(y) := \inf \left\{ z \in IR \mid F(z) = y \right\}, \quad y \in [0,1].
\]

In this case – depending on the type of the desired distribution – the values \( \pm \infty \) can also be assumed. Creating the desired random numbers is now very simple:

**Step 1:**

Create \( N \) independent, over \([0,1]\) evenly distributed random numbers \( y_1, \ldots, y_N \).

**Step 2:**

Transform \( y_1, \ldots, y_N \) zu \( x_1, \ldots, x_N \) with \( x_i = \tilde{F}^{-1}(y_i) \).
The adoption of the distribution for the transformed random variables can be easily tested: Let $X = F^{-1}(Y)$, in which case $Y$ is equally distributed over $[0,1]$, it is true that
\[ P(X \leq x) = P(F^{-1}(Y) \leq x) = P(Y \leq F(x)) = F(x) \quad \forall x \in \mathbb{R}, \]
i.e. $X$ exhibits the specified distribution function $F$.

**Example: Exponentially distributed random numbers**

From $F(x) = (1 - e^{-\lambda x})1_{[0,\infty)}(x)$ it follows that $F^{-1}(y) = -\frac{1}{\lambda} \ln(1-y), \; y \in (0,1)$, and one can apply the above-mentioned algorithm in order to create Exp(\(\lambda\))-distributed random numbers for $\lambda > 0$.

In case of the normal distribution the generalized inverse function of the distribution function is unfortunately not given in a closed analytical form, so that one very often favours another method. There are however (extremely complicated) functions which well approximate the inverse function of the distribution function of the normal distribution. The search for functions which simulate the desired inverse functions in a better way is incidentally a current topic in mathematical research because the random numbers created in such a manner are often better suited for multi-dimensional simulations with a high precision demand. (\(\rightarrow\)Ex.5.21)

**iii) Normally distributed random numbers**

For creating normally distributed random numbers, the so-called Box-Muller transformation has proven itself useful in general. The particular thing is that in doing so the random numbers are always created as pairs. If $Y$ and $Y'$ are independent and each equally distributed over $(0,1]$, $X$ and $X'$ are given as
\[ X = \sqrt{-2 \ln(Y)} \cdot \cos(2\pi Y'), \quad X' = \sqrt{-2 \ln(Y)} \cdot \sin(2\pi Y'), \]
independent and each $N(0,1)$-distributed. One firstly creates the independent, equally distributed random numbers and subsequently transforms them in pairs by means of the Box-Muller transformation into the random numbers distributed according to the standard distribution. If we need random numbers $Z \sim N(\mu, \sigma^2)$, we transform additionally:
\[ Z = \sigma \cdot X + \mu. \]

**Simulation of stock prices in the n-period binomial model**

By means of the possibility to create discretely distributed random numbers one can now simulate stock prices in an $n$-period binomial model. We will observe an $n$-period binomial model with time horizon $T$, modification factors $u$ and $d$, as well as modification probability $p$. One firstly creates $n$ independent, Bernoulli-distributed random variables $x_1, x_2, \ldots, x_n$ from $n$ independent, over $[0,1]$ equally distributed random numbers $y_1, y_2, \ldots, y_n$:
\[ x_i = \begin{cases} 
1, & \text{if } y_i < p \\
0, & \text{if } y_i \geq p, \; i=1,\ldots,n,
\end{cases} \]
in which case $p$ is the probability of a „success“ in the Bernoulli experiment. We will establish that $x_i = 1$ means that the stock price changes at time $t_i = iT/n$ at the specified factor $u$; $x_i = 0$ means that the stock price in $t_i$ changes at the factor $d$. The sum $x_1 + \ldots + x_i$ therefore represents the total number of upward movements of the stock up to the time point $t_i$. One thus obtains from the opening stock price $p_1$ a simulated stock price $P_i(t_i)$ in the binomial model as
\[ P(t_j) = p_1 \cdot u^{x_1 + \ldots + x_i} \cdot d^{-i(x_i + \ldots + x_j)} \]

for all time points \( t_1, t_2, \ldots, t_r \).

In order to make sure that there is no arbitrage, one selects \( d < e^{r \frac{T}{n}} < u \), in which case \( r \) is an appropriate market interest.

A typical result of such a simulation is presented in Chart 5.11, as parameters \( n=30 \), \( T=30 \), \( p=0.65 \), \( u=1.02 \), \( d=0.95 \), \( p_1=100 \) were chosen:

![Stock Price Chart](chart5.13)

**Chart 5.13 Simulation of the stock price in the binomial model**

In order for all the time points to have simulated stock price at their disposal, as a rule in the chart one connects the simulated values \( P(t_j) \) with a straight line. Seen in a calculational way, linear interpolation results in these interim values.

(→Ex.5.22, Ex.5.23)

**Simulation of stock prices in the Black-Scholes model**

Although the Black-Scholes model is conceptually very demanding and also as time continuous model would have to be simulated in uncountably many time points, it can still be simulated adequately and simply in a very exact way.

Reminder: The stock price \( P_1(t) \) at time \( t \) in the Black-Scholes model has the following form

\[ P_1(t) = p_1 \cdot e^{(b - \frac{1}{2} \sigma^2) t + \sigma W(t)} \]

Here only \( W(t) \) is accidental and it is true that

\[ W(t) \sim N(0,t) \]

If we want to merely simulate the stock price at a *fixed* time point \( t \), we need only the random number \( X \), distributed according to standard distribution. After multiplication by \( \sqrt{t} \) it is then true that

\[ \sqrt{t} \cdot X \sim N(0,1) \]

By inserting the random value \( \sqrt{t} \cdot X \) instead of \( W(t) \) we get that desired simulated stock price at time \( t \).
If one however wants to simulate the whole price development of $P_1(t)$ for $t \in [0,T]$, one selects an adequately refined timing of the interval $[0,T]$. In this way e.g. for $T=1$ the fragmentation into $N=500$ intervals of the length $T/N=0.002$ is, based on experience, perfectly sufficient. Subsequently one creates the stock price $P_1(t_i)$ in time points $t_i = i \cdot T/N$, $i=1,2,\ldots,N$, and connects the points created in such a way through linear interpolation. For this purpose one proceeds as follows (this algorithm is very suitable for programming):

Algorithm: Simulation of the stock price in the Black-Scholes model

1. Set $P_1(0)=p_1$, $W(0)=0$, $t_0=0$, $i=0$.
2. For $i=1$ to $N$ repeat

   $$ t_i = i \cdot \frac{T}{N} .$$

   Create the random number $X_i$ distributed according to standard normal distribution, which is independent from the previously created numbers and by means of that calculate:

   $$ W(t_i) = W(t_{i-1}) + \sqrt{\frac{T}{N}} \cdot X_i ,$$

   $$ P_1(t_i) = p_1 \cdot \exp \left( \left( b - \frac{1}{2} \cdot \sigma^2 \right) \cdot t_i + \sigma \cdot W(t_i) \right) .$$

3. Connect the points $P_1(t_0), \ldots, P_1(t_N)$ linearly.

As we can see, creating stock prices in the Black-Scholes-Modell is no longer difficult and we obtain, in case of a sufficiently big choice of $N$, the price developments typical of the Black-Scholes model as simulation, as for example represented in Chart 5.12. We observed the model with time horizon $T=30/360$, with number of simulation steps $N=30$, volatility $\sigma=0.3$, medial rate of return $b=0.05$, as well as opening price $p_1=100$.

![Chart 5.14 Simulation of the stock price in the Black-Scholes model](chart)

(→Ex.5.24, Ex.5.25)
More realistic $n$-period binomial models

If one compares Chart 5.12 with Chart 5.11 it seems that the multi-period binomial model is completely inadequate for simulating stock prices. The development in Chart 5.12 corresponds much rather to a 30-day stock chart which we find in the business paper. However one can also create realistic price development by means of a binomial model, if one selects the timing precisely enough and adapts the parameters well.

In order to approximate a Black-Scholes stock price with volatility $\sigma$ and medial rate of return $b$, in practice one often selects

$$ u = 1 + \sigma \sqrt{\frac{T}{n}}, \quad d = 1 - \sigma \sqrt{\frac{T}{n}}, \quad p = \frac{1}{2} + \frac{b \sqrt{T}}{2\sigma}. $$

A justification for the selection of $u$ and $d$ arises from the Taylor expansion of the first order of the exponential function for the selection of $u$ and $d$ in the section on the connection between the binomial and Black-Scholes model. In order to obtain useful results, one has to select a very big number of periods $n$! By means of this parameter selection and the values $T = 30/360$, $n = 300$, $\sigma = 0.3$, $b = 0.05$, $p_1 = 100$ we get the picture in Chart 5.13.

![Chart 5.15 Simulation of the stock price in the 300-period binomial model](image)

Exercises

**Ex.5.21** Observe the random variable $X$, equally distributed over $[-1,1]$.

a) Indicate the density of the random variable $X$!

b) Calculate the distribution function of $X$! Note: If the random variable $X$ features density $f$, the probability is calculated as

$$ P\{X \leq c\} = \int_{-\infty}^{c} f(x) \, dx. $$

c) How can one create a random variable with exactly this distribution from a random variable which is equally distributed over $[0,1]$?

**Ex.5.22** Simulate a stock price in the 10-period binomial model with parameters $T=1$, $p=0.55$, $u=1.05$, $d=0.98$, $p_1=100$! Use the following random numbers, equally distributed over $[0,1]$:

0.2, 0.44, 0.8, 0.1, 0.13, 0.96, 0.7, 0.35, 0.4, 0.28!

Draw up a stock chart using these data!

**Ex.5.23** Simulate a stock price in the 15-period binomial model with parameters $T=1$, $p=0.333$, $u=1.1$, $d=0.98$, $p_1=100$! Use a dice to create random numbers! Document the course of your simulation and finally draw a stock chart!
Ex.5.24 Simulate a stock price in the Black-Scholes model with parameters \(N=10\), \(\sigma=0.2\), \(b=0.08\), \(p_1=100\)! Use the following random numbers, distributed according to standard normal distribution: 0.2, -0.44, 0.8, 1, 1.3, -2.96, 0.7, 0.35, 0.4, 1.28!

Draw up a stock chart based on these data!

Ex.5.25 Simulate a stock price in the Black-Scholes model with parameters \(N=15\), \(\sigma=0.4\), \(b=0.1\), \(p_1=100\)!

Use the following random numbers, equally distributed over \([0,1]\):

0.2, 0.44, 0.8, 0.1, 0.13, 0.96, 0.7, 0.35, 0.4, 0.28 0.63 0.27 0.84 0.71 0.14!

Create a stock chart using these data!

Ex.5.26 Calculate the parameters \(u\) and \(d\), as well as \(p\), which were used to create Chart 5.13!

Ex.5.27 Simulate a stock price in the 15-period binomial model for a stock with parameters \(\sigma=0.25\), \(b=0.09\), \(p_1=100\)!

Select \(u\) and \(d\), as well as \(p\), appropriately!

Ex.5.28 Choose an interesting stock from everyday life, search the Internet for its volatility \(\sigma\) and the medial rate of return \(b\)!

Simulate the stock price by means of these specifications by using different models for the following 5 days and then compare your result with the actual subsequent stock price development of this stock! If you have no random numbers at hand, create appropriate random number by means of a dice or a coin! Document your simulations and observations accurately!

5.10 Summary

In this chapter we introduced the \(n\)-period binomial model and the Black-Scholes model for the purpose of modelling a stock price. In the \(n\)-period binomial model the stock price changes at an \(n\) specified time points with probability \(p\) for the factor \(u\) and with probability \((1-p)\) for the factor \(d\), respectively. Therefore after \(n\) periods one obtains the stock price of

\[ P_1^{(n)}(T) = p_1 \cdot u^X \cdot d^{(1-X)} , \]

in which case \(p_1\) is the price of the stock in \(t=0\) and \(X\) depicts the number of stock price modifications for the factor \(u\).

In the Black-Scholes model the stock price at each time point \(t\in [0,T]\) is modelled as

\[ P_1(t) = p_1 \cdot e^{(b-\frac{1}{2}\sigma^2)t + \sigma W(t)} \]

in which case \(b\) is designated as the medial rate of return of \(P_1(t)\) (note: it is true that \(E(P_1(t))=p_1 e^{b \cdot t}\)) and \(\sigma\) as the volatility of \(P_1(t)\). \(W(t)\) is a stochastic process, which in particular fulfils

\[ W(t) - N(0,t) \]

or the so-called (one-dimensional) Brownian movement. Further features, such as no-arbitrage condition of the market, which consists of the fixed deposit investment and the particular stock, as well as algorithms for simulating the above stock price models, is an important element of this chapter.

5.11 Outlook: Newer price models

Although the Black-Scholes model reflects the reality of the stock price movements very well and is accepted in practice, it also has its weaknesses, which inspired the development of even more complex and realistic models for stock prices. As a matter of fact these models are not meant to
be introduced here, but we should at least indicate a few reasons for the necessity of improving the Black-Scholes model.

**i) Independency of the relative price accretion**

Based on the empirical data it is easy to check whether the rates of return of stock prices (meaning the relative accretion) are as a rule almost correlated, but not independent.

**ii) Constant volatility**

The assumption of constant volatility would have to express itself in practice in such a way that the daily rates of return of the stock prices over time are equally distributed. Instead one, however, often observes the so-called „volatility clustering“, i.e. big upward deflections follow big downward deflections (and vice versa), while at the same time there are phases in which only small values of the rate of return are observed. This also becomes clear when looking at Chart 5.14 where the daily rates of return \( \frac{S(t+1) - S(t)}{S(t)} \) of the price of the Deutsche Bank-stock are cleared away. This is why the price models with so-called stochastic volatility were developed. More specifically: the constant \( \sigma \) in the Black-Scholes model is replaced by an appropriate stochastic process which behaves as a further Brownian movement.

**iii) Lognormally distributed rates of return**

If one creates a histogram of the logarithmic rate of return \( \ln\left(\frac{S(t+1)}{S(t)}\right) \) of the stock prices, one can notice that one typically observes more very small and very big values than in case in which the normal distribution is available. For this reason in the previous years there has been intensified search for distribution which has the capacity of explaining this observed performance on the market better than the normal distribution. Hyperbolic distributions and \( t \)-distributions proved themselves in this case as suitable candidates. An elaboration on the background of the belonging price processes exceeds the technical limits of this manuscript. However, we will still hold constant that the Black-Scholes model, despite certain shortcomings, plays an outstanding part in theory as well as practice of financial mathematics, and is therefore considered a benchmark for newly-developed models.

![Chart 5.16 Daily rates of return of the Deutsche Bank-stock from 31.10.97-31.10.98](image-url)